Infinite Series

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1. Concept and distinctions

In general, an infinite series is a series without an ending term, like the series of natural numbers, 1, 2, 3… Mathematician and logician Georg Cantor, defined it more precisely as a series that has the same number of terms as one of its subseries. For example, the series 1, 2, 3…, the series of natural numbers, has a subseries 2, 4, 6…, the series of even numbers. However, there are just as many even numbers as there are natural numbers, paradoxical as that may sound. That paradox identifies the series as infinite.

Among infinite series, we may distinguish between *actual* and *potential* infinites. The set of natural numbers is an actual infinite: that set actually contains an infinite number of members. A potential infinite, however, is a series that approaches an infinite number but never reaches that point, as when we try to list all the natural numbers, one by one, or when we divide an object by half, and then by half again, and so on. In those cases we never reach an ending point, a last member of the series. We never reach a number that we could call "infinity."

2. Apologetic Importance

Some forms of the cosmological argument for the existence of God deny the existence of certain kinds of infinite series. Thomas Aquinas, in the first three of his "five ways," denied that chains of causes (causes of motion, being, and necessity, respectively) can go back forever. He argued that every causal chain has a beginning: a first mover, a first cause of being, and a first necessary being, namely God. (See Aquinas, *Summa Theologica,* Part 1, Question 2, Article 3.) The *Kalam* argument of Al-Ghazali, recently expounded by William Lane Craig, denies that there can be an actually infinite series of events succeeding one another in time. Therefore, the universe had a beginning, which must be explained by a divine cause.

Craig argues, first, that there cannot be an actually infinite collection of *things* (though there can be actually infinite sets of numbers), and, second, that even if such a collection were possible it could not be achieved by adding one member after another, as must happen in a temporal succession of events.

To show that there cannot be an actually infinite collection of things, he refers to the paradoxes noted by Cantor: (1) In an infinite series, the whole is equivalent to some of its parts. (2) One can add members to an infinite set without increasing the number of members in the set. (The number remains at infinity.) (3) One can remove members from the set without decreasing its membership. Such is the case in the abstract world of numbers. But, Craig says, it would be impossible to

have a set of concrete objects or a series of events that had these properties. He uses the illustration of "Hilbert's Hotel" from George Gamow, *One, Two, Three, Infinity*, p. 17: If a hotel had an infinite number of rooms filled with guests, additional guests could check in without anyone moving out, and the number of guests would be the same as before. The sign could read, "NO VACANCY— GUESTS WELCOME" (Craig, *Reasonable Faith*, p. 96).

Then Craig argues that even if we grant the possibility of an actually infinite collection of things, we cannot form such a collection by adding one member after another. It is impossible, for example, to count an infinite collection one by one; for "No matter how many numbers you count, you can always add one more before arriving at infinity" (p. 98). The same must be said of an infinite series of events in time. If the process of nature and history extends infinitely far into the past, then it is an infinite succession of events, and that succession has proceeded one by one, ending precisely at the present moment. But why did it end now, rather than yesterday, or a thousand years ago? For on this hypothesis yesterday was also the end of an infinite chain of events, and so was the moment a thousand years before the present one. But, in fact, there can be no end at all, for an infinite series never ends. So, Craig concludes, the series of past events is finite. Therefore the universe had a beginning, and therefore a cause, because "whatever begins to exist has a cause" (p. 92).

3. Evaluation

Certainly it is difficult to conceive of an actually infinite collection of things. Hilbert's Hotel is counter-intuitive; but many find the Cantor paradoxes themselves hard to believe at first hearing. After we learn to work with infinite sets of numbers, we tend to accept the Cantor definitions as a matter of course. We have not, however, encountered infinite sets of material objects. But if we ever do, might we not eventually get used to the strange properties of such sets? Here, images are important. The idea of an infinite hotel is somewhat ridiculous, as is, say, the idea of a hotel with the hiccups. But how about the idea of an infinitely extended chain of beads? Might we not one day get used to the idea of adding or subtracting beads without changing the number in the infinite collection?

Part of the problem is that when we try to picture in our minds an infinite hotel, we tend to think of it as a finite hotel with very odd properties: people being squeezed into it without others being squeezed out. But if the hotel were truly infinite, those properties would not be odd, but expected, hard as it may be to imagine these properties in a mental picture. It is also hard to imagine such properties in a series of numbers; but Cantor proved that they exist.

Similarly, the notion of an infinite series of events continuing through time is hard to comprehend, but is it impossible? I agree with Craig that it is impossible to count through an infinite series and end with a final number. But (1) if time itself

were subjective, rather than objective, then an infinite set of past events might exist simultaneously (like the series 1, 2, 3...), rather than existing by a temporal process of addition. (2) The same would be the case if time were an objective dimension of n-dimensional space, and all events of past, present, and future, could be viewed together by a being of a higher dimension. And (3) if we could go backward in time from the present, then we could visit yesterday, the day before yesterday, and the day before that, much as we now move from today, to tomorrow, to the day after. In that case, we would perceive the days of past history much as we now perceive the days of the future: as a potential infinity, rather than an actual infinity.

Of course, these three suppositions are contrary to Craig's own theory of time. See his *Time and Eternity: Exploring God's Relationship to Time* (Wheaton, 2001). So I am not here challenging the consistency of Craig's view. But these considerations indicate that our present questions about infinite series do not have obvious answers. Indeed, they are linked to other issues that deserve book-length treatment.

Thomas Aquinas would object to supposition (3) that even a potentially infinite series of natural events in the past is insufficient to account for the world as we know it. For on this supposition, each event is caused by a previous one; no event actually begins the series. Therefore, no event (or group of events, by the same logic) serves as the cause of the rest. So the universe is uncaused, unexplained. Aquinas believes the universe must have a cause; so the chain of causal explanation cannot be infinite, even potentially infinite.

Aquinas argues that the universe has a cause; therefore there cannot be an infinite series of causes. Craig argues the reverse: there cannot be an infinite series of causes; therefore the universe must have one cause. I confess that I find Aquinas more persuasive: it seems more obvious to me that the universe requires a cause than that an infinite series of events is impossible. But even Aquinas's view requires assumptions, namely that nothing exists or happens without a sufficient cause, and that causes (including the cause of the universe) are accessible to human reason. Many skeptics of the past and present would not grant those assumptions.

My conclusion is that our concepts of cause, reason, and infinite series depend on worldviews, on ontological and epistemological assumptions. They are insufficient in themselves to serve as grounds for worldviews. A Christian theist will think differently from a skeptic about these matters. His Christian theism will govern his concepts of cause, reason, and infinity, rather than the reverse.

Bibliography

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